Semidirect Products Explained

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November 4, 2021

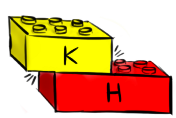
1 Introduction

A favorite activity of group theorists is to look at...well...groups. What groups exist? Which groups are the same and which are different? This isn't too far from children exploring what they can make from a Lego box they have at home. Of course, we can start by just trying to construct groups directly from the definition. This may get us to find the trivial group, $\mathbb{Z}_2$, or maybe even $\mathbb{Z}_p$. With some more searching, we might find that familiar objects like $\mathbb{Q}$ or $\mathbb{Z}[X]$ also fit the definition. Once we figure out that the Lego pieces can be combined to form more complex structures, however, our options become significantly more numerous than just digging through the box and searching for cool-looking pieces.

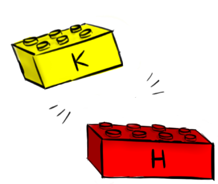
One simple way to combine two groups into another is to take the direct product. Let's say we have two groups K and H. We can define $G = K \times H$ to be a group whose elements are pairs (k, h) where $k \in K$ and $h \in H$, along with the group operation

$$(k_1, h_1)(k_2, h_2) = (k_1k_2, h_1h_2).$$

This works since $k_1k_2 \in K$ and $h_1h_2 \in H$. Note that the two pieces are neatly embedded in their product because $\{(k, 1) : k \in K\} \cong K$ and $\{(1, h) : h \in H\} \cong H$.

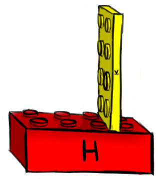


We can also look at direct products from the other direction. If a group has the right structure, then we can disassemble it into a direct product of two groups. More precisely, let G be a group and let H, K be two of its normal subgroups such that $H \cap K = \{1\}$ and $HK = G$. Then, we can show[1] that G is isomorphic to $H \times K$.



[1]Show that the map $\phi : G \to H \times K$ where $\phi(hk) = (h, k)$ is a well-defined isomorphism.

So, we have learned to combine our lego blocks and take them back apart. However, this is only one way to combine blocks. There are also less known methods.



Similarly, semidirect products are just another way to combine and disassemble groups. They will help us analize familiar groups and create ones we haven't seen before.

2 Semidirect Products

In a direct product $G = H \times K$, both H and K are normal in G. Semidirect products are a relaxation of direct products where only one of the two subgroups must be normal.

Let H and K be subgroups of G where $H \triangleleft G$ and $K \cap H = 1$ such that $HK = G$. Note that K need not be normal in G. We can see that for any $g \in G$, there is some $h \in H$ and $k \in K$ such that $hk = g$. Suppose there are two choices for h and k. We see that $h_1k_1 = g = h_2k_2$ implies that $h_1{h_2}^{-1} = k_1{k_2}^{-1}$, which means that $h_1{h_2}^{-1} = 1 = k_1{k_2}^{-1}$ because $H \cap K = \{1\}$. This means that $h_1 = h_2$ and $k_1 = k_2$, so the choice of k and h is actually unique. So, every pair of elements (h, k) where $h \in H$ and $k \in K$ corresponds to exactly one element of G. This smells a lot like the direct product, leading us into an attempt of defining a semidirect product.

Definition 1 (first attempt): Let G have subgroups H and K where $H \triangleleft G$ and $HK = G$ and $H \cap K = \{1\}$. We say that G is a semidirect product of H and K with H as the normal subgroup. This is written as $G \cong H \rtimes K$. $\triangleleft$

Unfortunately, we'll later see that by this definition it's possible to have $G \cong H \rtimes K$ and $G' \cong H \rtimes K$, but $G \ncong G'$, so we need to give some more information about the group to make our definition correct.

In direct products, we know that $khk^{-1} = h$, but here we can't deduce that since K may not be normal. However, since H is normal, for any $k \in K$ and $h \in H$, we have that $kh = h'k$ for some $h' \in H$. So, $khk^{-1} = h'kk^{-1} = h'$. This shows that K acts on H by conjugation. In two different groups, this action could be different, but H and K would be the same. How K acts on H is exactly the missing information that we need to add. Let's try again.

Definition 1: Let G have subgroups H and K where $H \triangleleft G$ and $HK = G$ and $H \cap K = \{1\}$. Let $\varphi : K \to \mathrm{Aut}(H)$ be that conjugation action of K on H. We say that G is a semidirect product of K acting on H. This is written as $G \cong H \rtimes_\varphi K$. $\triangleleft$

Let's see if the additional information fixed our definition. For any $g_1 = h_1k_1$ and $g_2 = h_2k_2$ in G,

$$g_1g_2 = h_1k_1h_2k_2 = h_1k_1h_1{k_1}^{-1}k_1k_2 = h_1\varphi(k_1)(h_2)k_1k_2 \in HK$$

where $h_1\varphi(k_1)(h_2) \in H$ and $k_1k_2 \in K$. So, G is unique for a given H and K, meaning our definition is now correct!

We want to be able to look at semidirect products both from below and from above. In other words, we want to know how to build a semidirect product from two groups, but also be able to recognize when we can decompose a group into a semidirect product of two smaller groups, like what we did with direct products. We already did the latter with definition 1, so now let's do the former. Let's construct the semidirect product from H, K and how K acts on H.

Definition 2: Let H and K be groups and let K act on H. Let $\varphi : K \to \mathrm{Aut}(H)$ be the action of K on H. Let G be a group of all pairs (h,k) with $h \in H$ and $k \in K$ with the group operation

$$(h_1,k_1)(h_2,k_2) = (h_1\varphi(k_1)(h_2), k_1k_2).$$

Denote G by $H \rtimes_\varphi K$. $\triangleleft$

It's easy to verify that G is a well-defined group with $(\phi(k)^{-1}(h^{-1}), k^{-1})$ as the inverse of (h,k). The claim is that this definition is aligned with our earlier one.

2.1 Equivalence of Definitions

We need need to show that our two definitions of semidirect products are equivalent. We must go from definition 1 to definition 2 and back.

The forward direction is easier, so let's do it first.

Theorem 1: Suppose we can disassemble G as $G \cong H \rtimes_\varphi K$ according to definition 1. Let $G' = H \rtimes_\varphi K$ according to definition 2. Then, $G \cong G'$. $\triangleleft$

The proof is just creating an isomorphism between G and G'.

Proof of Theorem 1: Consider the map $\psi : G \to G'$ where $\psi(hk) = (h,k)$. As we have discovered earlier, each element of G corresponds to exactly one pair (h,k), so ψ is a bijection. We can see that

$$h_1k_1h_2k_2 = h_1k_1h_1{k_1}^{-1}k_1k_2 = h_1\varphi(k)(h_2)k_1k_2,$$

so

$$\psi(h_1k_1)\psi(h_2k_2) = (h_1,k_1)(h_2,k_2) = (h_1\varphi(k)(h_2), k_1k_2) = \psi(h_1\varphi(k)(h_2)k_1k_2) = \psi(h_1k_1h_2k_2).$$

This means that ψ is an isomorphism so $G \cong G'$. $\square$

Now let's go from definition 2 to definition 1.

Theorem 2: Let $G = H \rtimes_\varphi K$ according to defintion 2. Also, let $\hat{H} = \{(h,1) : h \in H\}$ and $\hat{K} = \{(1,k) : k \in K\}$. Then,

1. $\hat{H} \cong H$ and $\hat{K} \cong K$.
2. $\hat{H} \triangleleft G$.
3. $\hat{H} \cap \hat{K} = \{1\}$
4. For any $(h,1) \in \hat{H}$ and $(1,k) \in \hat{K}$, we have $(1,k)(h,1)(1,k)^{-1} = (\varphi(k)(h), 1)$.

So, $G \cong \hat{H} \rtimes_\varphi \hat{K} \cong H \rtimes_\varphi K$ according to definition 1. $\triangleleft$

This proof is also done without any tricks.

Proof of Theorem 2: We can go through the statements in order.

1. This is easy to see because $(h_1,1)(h_2,1) = (h_1h_2,1)$ and $(1,k_1)(1,k_2) = (\phi(1), k_1k_2) = (1, k_1k_2)$.
2. For any $(\bar{h},1) \in \hat{H}$ and $(h,k) \in G$, we see that

   $$(h,k)(\bar{h},1)(h,k)^{-1} = (h,k)(\bar{h},1)(\phi(k)^{-1}(h^{-1}), k^{-1}) = (h', kk^{-1}) = (h',1) \in \hat{H}.$$

   For some $h' \in H$. Note that we do not have to actually compute the product above. All we need is that the second component of the resulting tuple is 1.

3. This follows directly from the definition of $\hat{H}$ and $\hat{K}$.

4. Let $(h,1) \in \hat{H}$ and $(1,k) \in \hat{K}$. Then,

$$(1,k)(h,1)(1,k)^{-1} = (1,k)(h,1)(1,k^{-1}) = (1,k)(h\varphi(1)(1),k^{-1}) = (1,k)(h,k^{-1}) = (\varphi(k)(h),1),$$

which is what we wanted.

Since $\hat{H}$ and $\hat{K}$ fit all of the conditions of definition and φ is the conjugation action, we have that $G \cong \hat{H} \rtimes_\varphi \hat{K} \cong H \rtimes_\varphi K$. $\square$

With this, we have shown that definition 1 and definition 2 are equivalent!

2.2 A Special Case

Since we said that semidirect products are a relaxation of direct products, direct products must be a special case of semidirect product. Indeed, the semidirect product is just the direct product if K and H both happen to be normal. Equivalently, this happens if the action is trivial. Let's prove this.

Theorem 3: Let H and K be groups and let φ be an action of K on H. The following are equivalent:

1. The identity map $Id : H \rtimes_\varphi K \to H \times K$ is an isomorphism.
2. The action φ is trivial.
3. $\{(1,k) : k \in K\} \triangleleft H \rtimes_\varphi K$.

So, a direct product is a special case of a semidirect product. $\triangleleft$

The proof is a straightforward shuffling of definitions.

Proof of Theorem 3: We can prove the equivalence in four parts (2) $\Leftrightarrow$ (1) $\Leftrightarrow$ (3).

- (1) $\Rightarrow$ (2). By assumption, we can multiply elements of $H \rtimes_\varphi K$ coordinate-wise. So, for any $h \in H$ and $k \in K$, we see that

$$(\phi(k)(h),k) = (1,k)(h,1) = (h,k),$$

so $\phi(k)(h) = h$. This means that ϕ is the trivial action.

- (2) $\Rightarrow$ (1). If φ is trivial, then

$$(h_1\phi(k_1)(h_2), k_1k_2) = (h_1h_2, k_1k_2),$$

so the identity map from $H \rtimes_\varphi K$ to $H \times K$ is an isomorphism.

- (3) $\Rightarrow$ (1). Let $\hat{H} = \{(h,1) : h \in H\}$ and $\hat{K} = \{(1,k) : k \in K\}$. We know $\hat{H}$ is normal in $H \rtimes_\varphi K$ by definition. If $\hat{K}$ is normal in $H \rtimes_\varphi K$, then by definition $H \rtimes_\varphi K \cong H \times K$ with the identity map as the isomorphism.

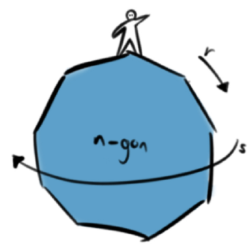
- (1) $\Rightarrow$ (3). If $H \rtimes_\varphi K \cong H \times K$ with the identity map, then $\hat{K} \triangleleft H \rtimes_\varphi K$ because $\hat{K} \triangleleft H \times K$.

With this, we are done. $\square$

3 Examples

Now that we have an idea for what semidirect products are, let's look at some concrete examples. Throughout this section, we denote the cyclic group of order n by C_n.

Example 1: Consider D_{2n}. Let $K \subseteq D_{2n}$ be the subgroup consisting of the two reflections and let $H \subseteq D_{2n}$ be the subroup of rotations. Note that $K \cong C_2$ and $H \cong C_n$. We can see that $H \triangleleft D_{2n}$ because H has index 2. Also $H \cap K = \{1\}$ and $HK = D_{2n}$. The conjugation action $\varphi : K \to \mathrm{Aut}(H)$ is $\varphi(1)(h) = h$ and $\varphi(s)(h) = h^{-1}$ where s is the nonidentity element of K. So, we can conclude that $D_{2n} \cong H \rtimes_\varphi K \cong C_n \rtimes_\varphi C_2$. $\triangleleft$



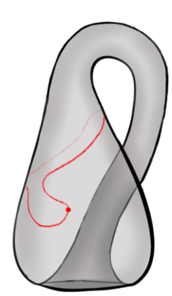
Now, let's actually construct something we haven't seen before. After all, one of the goals of semidirect products is to discover new groups.

Example 2: Let $H = \mathbb{Z}$ and $K = C_2$. Let $\varphi : K \to \mathrm{Aut}(H)$ to be the action of K on H where $\varphi(1)(h) = -h$, like in example 1. Then $H \rtimes_\varphi K$ is somewhat like the dihedral groups, but infinite. We call this D_∞.

Example 3: Out of the groups of order 12, we know of C_{12} and $C_6 \times C_2$ and the dihedral group D_{12}. Potentially, we have also seen the alternating group A_4. There is actually one more group of order 12 that we can find using semidirect products! Let g be the generator of C_4. The group is $C_3 \rtimes_\varphi C_4$ where $\varphi(g)(x) = x^{-1}$ for any $x \in C_3$. This is the same as the group generated by the elements g and h where $g^4 = h^3 = 1$ and $ghg^{-1} = h^{-1}$. With a little bit of work, it's possible to show that these are the only 5 groups of order 12. So, semidirect products help us classify all groups of order 12. $\triangleleft$

For the readers acquainted with fundamental groups, consider the following cute example.

Example 4: Let $\varphi : \mathbb{Z} \to \mathrm{Aut}(\mathbb{Z})$ be the homomorphism where $\varphi(n)(m) = (-1)^n m$. Then, $\mathbb{Z} \rtimes_\varphi \mathbb{Z}$ is the fundamental group of a Klein bottle. Though, the explanation is beyond the scope of this text. $\triangleleft$



4 Don't be Fooled

It's also important to note things that aren't true about semidirect products. Here are a few notable counterexamples. We say a semidirect product is nontrivial if neither of the two groups in the product is trivial.

Counterexample 1: Containing a proper normal subgroup is not enough to be a nontrivial semidirect product. Take $C_4 = \{1, g, g^2, g^3\}$, for example. We can see that $H = \{1, g^2\} \cong C_2$ is a proper normal subgroup. If $C_4 \cong H \rtimes_\varphi K$, then K must contan 1 and either g or g^3 because $HK = C_4$. However, then it also must contain $(g)^2 = (g^3)^2 = g^2$, which is impossible because K can only have 2 elements. So, no such K exists, so C_4 is not a semidirect product. There isn't a known way to characterize all semidirect product groups. $\triangleleft$

Counterexample 2: The same group may be written as multiple nontrivial semidirect products. For example, there exist homomorphisms φ and ψ such that, $C_2 \rtimes_\varphi D_{12} \cong D_8 \rtimes_\psi C_3$. In fact, there are two more ways to write this group as a nontrivial semidirect product. $\triangleleft$

Counterexample 3: Finally, we want to emphasize that the homomorphism φ we choose is important. Consider the two groups C_8 and C_2. There are four different things that $C_8 \rtimes_\varphi C_2$ could be. One of these is the direct product $C_8 \times C_2$, and another is D_{16}, as mentioned in example 1. $\triangleleft$

5 Acknowledgements

The ideas in this text are taken from H.E. Rose's *A Course on Finite Groups. (2009)* and Dummit & Foote's *Abstract Algebra (2004)*, as well as Richard Taylor's amazing *Groups and Rings* class at Stanford University.