A Role-Reversing Complete Waltz Mixer

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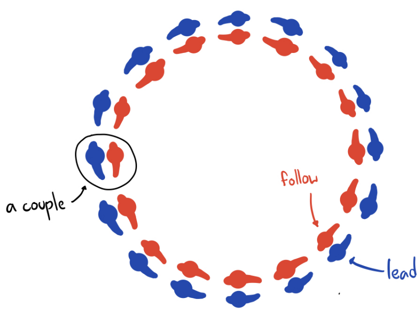
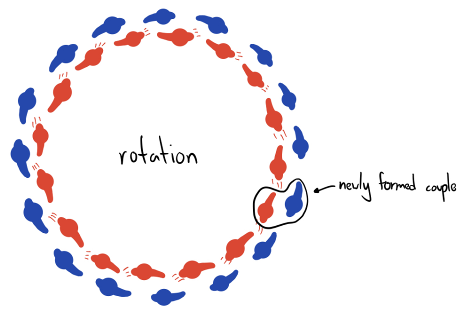
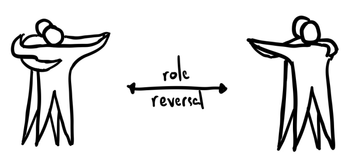
1 Introduction

I was taking a social dance class[1], and on the waltz unit, we sometimes learn something called a "waltz mixer". This is when all of the couples stand in a circle and follow a choreography that leads to switching partners. If repeated enough times, you end up dancing with everyone of the opposite role[2].





Waltz is a dance where it's very easy to switch roles, so it's possible for leads to become follows and follows to become leads in the middle of the dance. In class, we tested a waltz mixer where the roles are swapped every round, so you alternate between dancing as a lead and dancing as a follow. In this mixer, however, everyone still only gets to dance with those who were the opposite role in the beginning. You still dance with the same set of people as in the standard mixer, just sometimes role-reversed. Original leads never dance with other original leads, and original follows never dance with other original follows.



Since we are no longer restricted to dancing with the opposite role, the natural question is if it's possible to design a waltz mixer where everyone gets to dance with everyone else. In other words, does there exist a complete waltz mixer? The answer turns out to be surprisingly nontrivial—enough to make it the subject of this text. We'll see a better way to visualize this problem and explore various solutions.

We speak about waltz here, but really this is applicable in many situations. Maybe we have a group of

[1] DANCE 156 at Stanford.

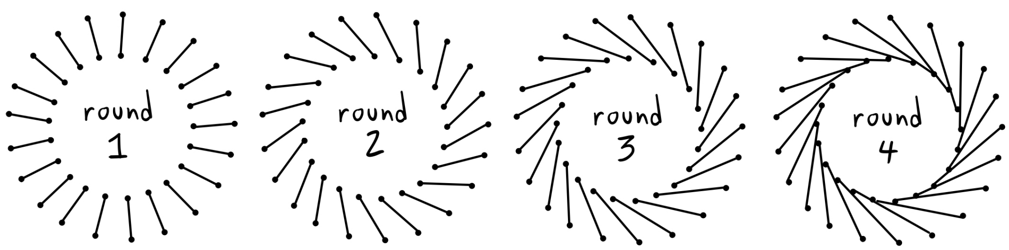
[2] Social dances usually have a lead and a follow role. Each couple consists of one lead and one follow.

friends and we want everyone to meet everyone one-on-one, but don't want anyone sitting out at any point in time, to make it as quick as possible. Maybe we have a set of servers on a network and we want every server to communicate with every other server in the least amount of rounds. It's all rewordings of the same problem. Though, with waltz, there are also some other practical restrictions—we don't want the transitions between rounds to be too crazy, so ideally the next partner is always someone who's next to you.

2 The Problem

To start, let's ignore the practical restrictions of waltz and assume that our dancers can teleport. We'll also assume that they have a perfect memory and the choreography can be arbitrarily complicated. Later, we'll look at how we can make our mixers practical.

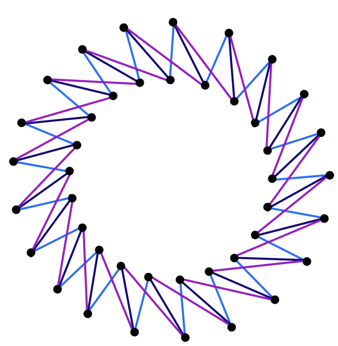
A reliably good way to think about partnering is graphs. Let's say we have n couples (so $2n$ dancers) participating in our mixer. We can represent each dancer with a vertex (i.e. point) in a graph and represent dancing together with edges (i.e. lines) connecting two vertices. So, for example, the first few rounds of the standard mixer without any role-reversing would look something like this:



In fact, the graphs for the rounds of the role reversing mixer we did in class would look the exact same.

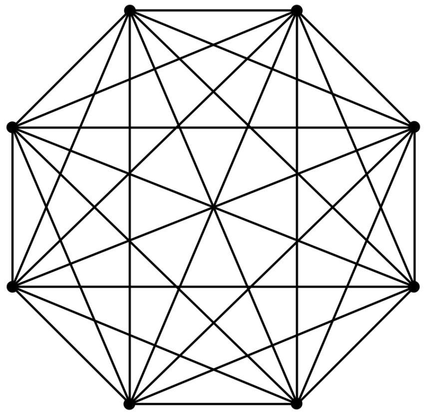
We can assign a color to each round to help us visualize this. We'll use this order of colors for the rest of this text:

As long as our choreography never has anyone dancing with the same partner twice, we can overlay the graphs for the rounds to create a colored graph for the entire mixer[3]. So, the first three rounds of the standard mixer would look something like this (we limit ourselves to only drawing 3 rounds to avoid a messy picture):

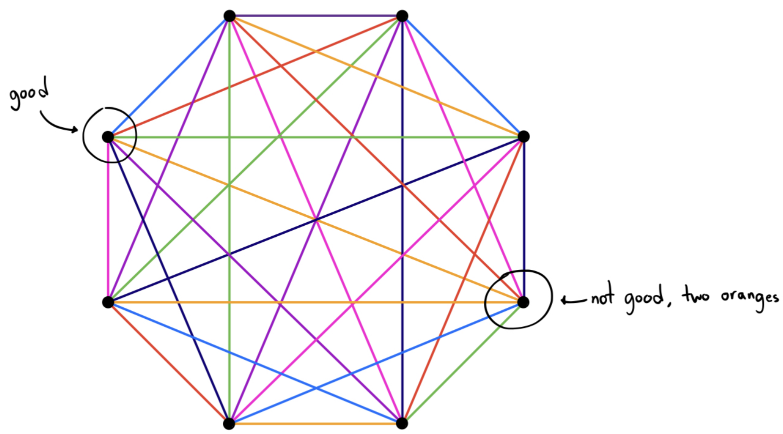


[3]For now, we'll set it as a goal that partners never repeat, but it's perfectly fine to have repeats in general. This can be represented by drawing two edges between the same two vertices.

If there are $2n$ dancers, then everyone has $2n-1$ other people they must dance with, so we will have $2n-1$ different colors. Our goal is for everyone to dance with everyone once, so the graph we want to color is one where every vertex is connected to every other vertex. This is called a complete graph on $2n$ vertices.



In the same round, the same dancer can't dance with two people, so we can't have two edges of the same color ever sharing a vertex. In other words, we want to color the edges of a complete graph so that each color appears only once at each vertex.



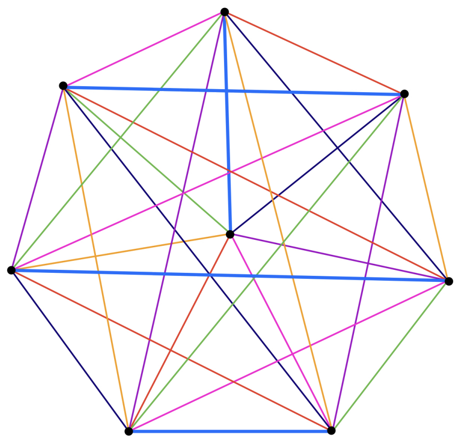
3 A Solution

In general, the number of colors it takes to color the edges of a graph with no two touching edges touching is called the *chromatic number* of the graph. There can be a whole book written about chromatic numbers of graphs, but we are concerned with specifically $2n$-vertex complete graphs[4].

It is known that the chromatic number of a complete graph with $2n$ vertices is $2n-1$. In other words, we can color a $2n$-vertex complete graph with $2n-1$ colors, which is exactly what we want to make our complete waltz mixer. I could only find one method of actually doing the coloring online. The construction of this coloring is as follows.

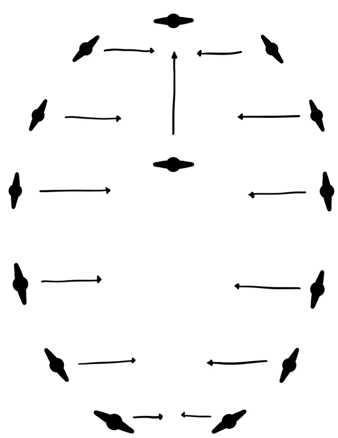
[4]By Vizing's theorem, the chromatic number of a graph is always either the maximum degree of the graph or one more than that. Determining which one is an NP-complete problem. This is just a side note—you do not need to understand what this means to read this text.

Construction 1: Select one special vertex and place it in the middle. Then, place the other $2n-1$ vertices in a circle around that special vertex. Next, color the edges going out of the center vertex with all $2n-1$ colors that we have. We'll call these the radial edges. For every edge that we haven't colored, find the radial edge that is perpendicular to it (there will always be one because the number of edges on the outside is odd), and color it the same color. ◁



We see that no two nonradial edges of the same color touch because they are all parallel, and the radial edges never touch other edges of the same color because they are perpendicular. So, this is a valid construction that we can turn into a complete waltz mixer!

You can imagine delegating a special person who stands in the middle. During each round, they select one person from the outside to dance with. The others find the person across from them in the circle to dance with. The circle squishes into a line for one round, then the dancers separate and the special person in the middle selects a new person to dance with. This is repeated until the special person has danced with everyone.

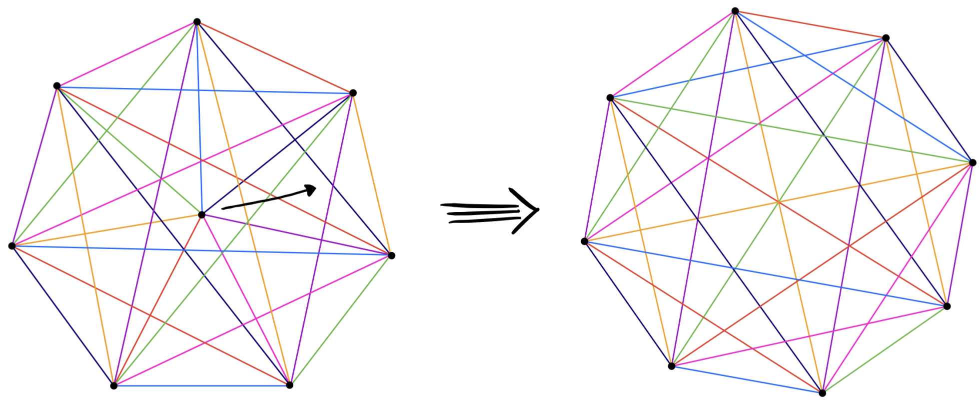


So we have a solution to the problem, but it's not a really good solution. In waltz mixers, we don't want to single out a specific dancer to be special. After all, social dance isn't a performance for one[5]. For the rest of this text, we'll look for a more practical construction of a complete waltz mixer.

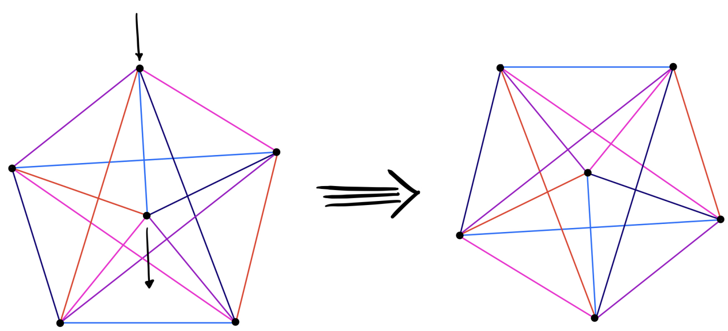
[5]Though, if we are servers looking to communicate, this solution is perfectly fine. If we are a group of friends at a birthday party, there is also an obvious selection of a special person.

4 A More Practical Solution

A natural attempt of avoiding singling out a dancer is to move them out of the middle. Unfortunately, the mixer scheme then becomes so complicated, that it's only feasable for teleporting dancers with a perfect memory.

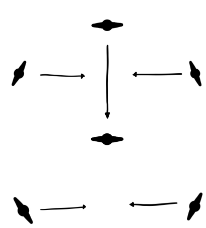


The next guess of what we could try is to change the person in the middle every round. For 3 couples (6 dancers), this actually works! With only 5 people on the outside circle, imagine moving a radial edge to the other side so that the person in the middle moves out and someone moves in. It turns out that the coloring still follows the rules of all nonradial edges being perpendicular to the radial edge of the same color.

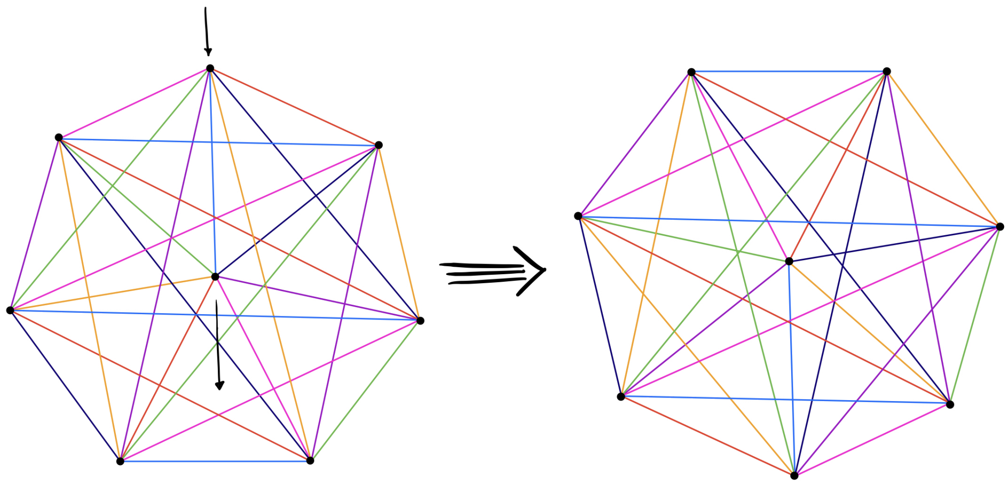


So, changing the person in the middle in this way gives us essentially the original construction, just with different people and a different order of colors.

So, you can imagine the following mixer for 3 couples. One person starts in the middle, the rest on the outside, like last time. The person in the middle picks someone they haven't danced with, the rest join the person across, like last time. However, this time, after the round ends, the person selected goes into the middle and the previous person in the middle joins the outer circle on the opposite side. This is repeated for 5 rounds, after which everyone has been in the middle and everyone has danced with everyone.



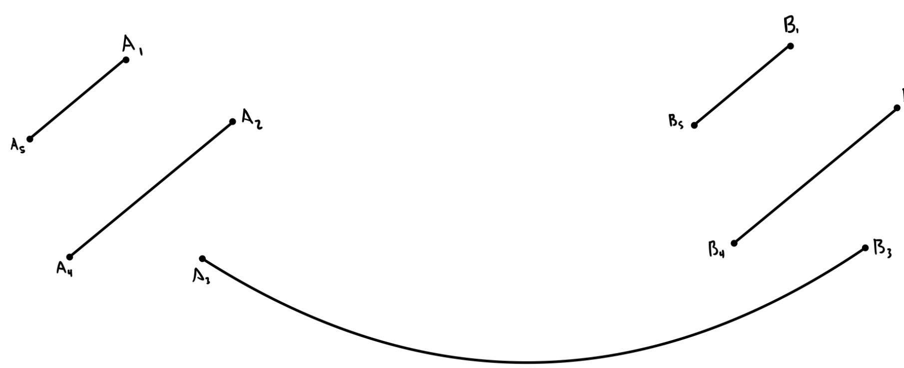
Unfortunately, the same idea doesn't work for more than 6 dancers. When we try to swap in the center dancer like last time, the other edges don't keep the nice property of being perpendicular to the radial edge of the same color. So, following the same scheme will no longer work[6].



To address this, we must think of an entirely new construction. Let's look at a construction that works well without needing to teleport, have an incredible memory, or select a star of the show. It only works for an odd number of couples, but we'll later see how to extend it to work for any number of couples.

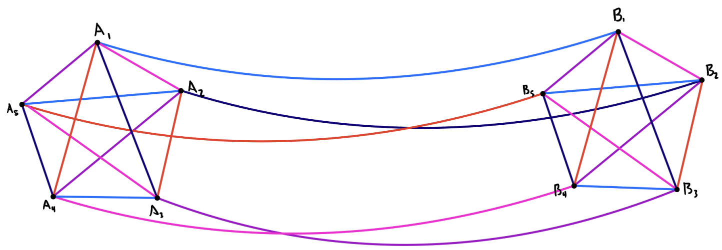
Construction 2: Suppose n is odd. We'll create a coloring of the $2n$-vertex complete graph with $2n-1$ colors. First let's split the vertices into two groups of n. The construction will be in two stages. First, we'll connect everything in each group within itself. Then we'll connect everything in one group with everything in the other. The first stage will take n colors and the second will take $n-1$ colors.

Let's call the vertices in the first group $A_1, A_2, \ldots A_n$ and the vertices in the second group $B_1, B_2, \ldots, B_n$. For simplicity, let's arange the vertices in each group in a circle. During the first stage, in round i, we color the edge from A_i to B_i as well as every edge that goes perpendicular to the radius created by A_i in the A group and perpendicular to the radius created by B_i in the B group, similar to our first construction.

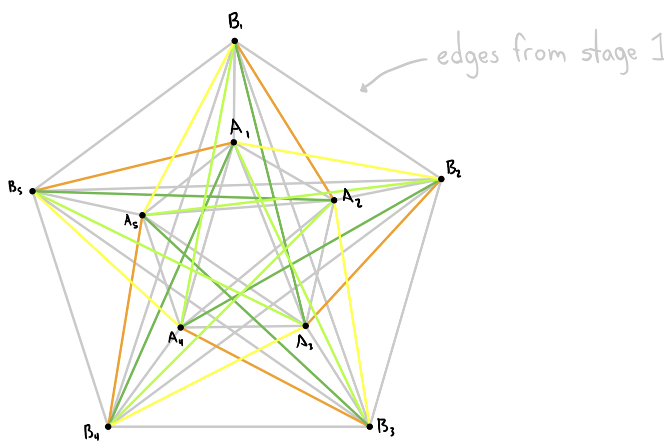


[6]In fact, no switching around of vertices will keep the nice property unless the center stays the same. Not even something more complicated than pushing the center out on the opposite side. So, the special person is indeed special. For mathematians, we can define isomorphisms of colorings and show that even with 4 couples, automorphisms preserve the center element. But, again, there is no need to be a mathematician to enjoy this text and the footnotes can be ignored.

After the first stage is done, we have used n colors, every edge within the groups has been colored, and there are n edges that go across groups that have been colored.



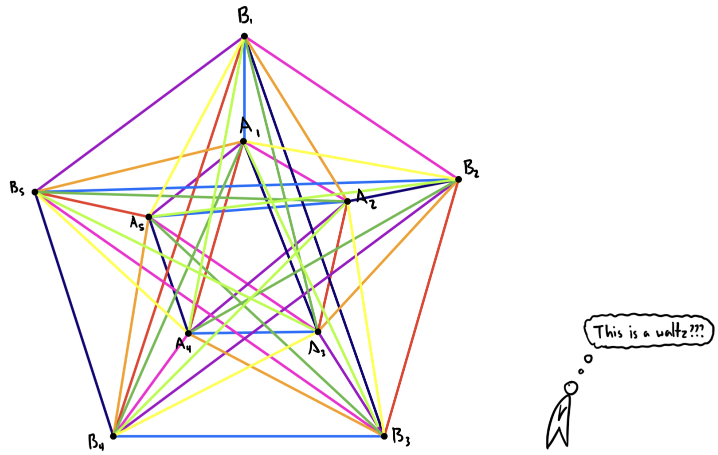
For the second stage, let's arange the groups in two concentric circles with one group on the inside and one on the outside, so that everyone faces the one person from the other groups that they have danced with. Now, we can just proceed like the standard waltz mixer, where each color connects to the next person in the other circle. We can skip the first person that they have already danced with, so we need $n-1$ colors for the second stage.



By the end of the second stage all edges have been colored, and we've used exactly $2n-1$ colors, as we wanted [7]. ◁

For 5 couples, the complete colored graph would look like this:

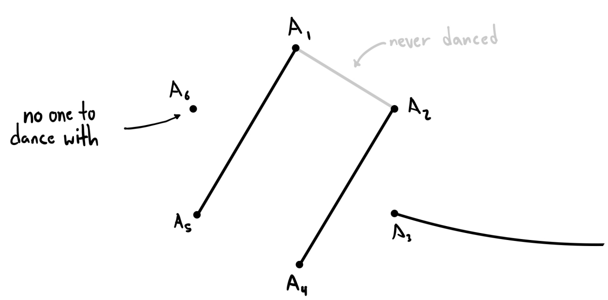
[7] Another note for mathematicians: This construction, is, in fact different from Consruction 1. For $n = 3$ couples, the graph from Construction 2 has a Hamiltonian path going only over edges of 2 colors, but the graph from Construction 1 does not.



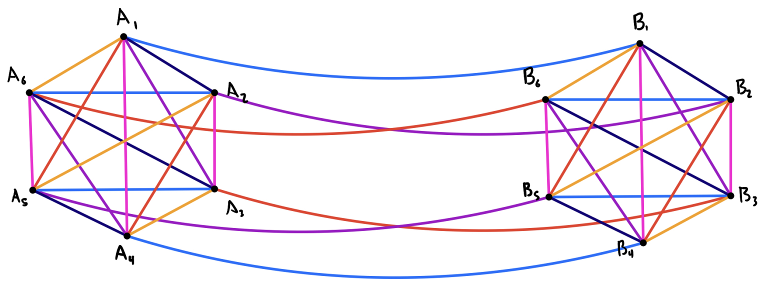
We can imagine a real dance corresponding to this construction. It'll be in two stages. First, the dancers split up into two groups. Each dancer has a special round when they get to dance with someone in the opposite group while the others dance within their groups. In the second stage, the dancers reorganize into two concentric circles and complete with the standard waltz mixer choreography.

I find this construction particularly nice because each dancer gets a moment when they are the special one, opening opportunities for an interesting show. The two stages can also be swapped without a problem to create a more dramatic effect by starting with a normal mixer and then diving into the coolness of the first stage. In fact, the stages can even be interwoven for a compliated, but interesting choreography. It wouldn't be hard to organize so that the special dancers in each round of the first stage are next to each other, so that the transitions are reasonable.

Note that this construction doesn't work if n is even because there will be a point directly opposite of A_i and B_i in their groups that don't have a dance partner on round i, and there are some edges within each group that are not perpendicular to any radius, so those dancers will never get to dance together.



One solution to this is to modify stage 1 to have two dancers cross to the other group in some rounds, and none in other. That way, we can cover all n edge directions within each group in n colors.



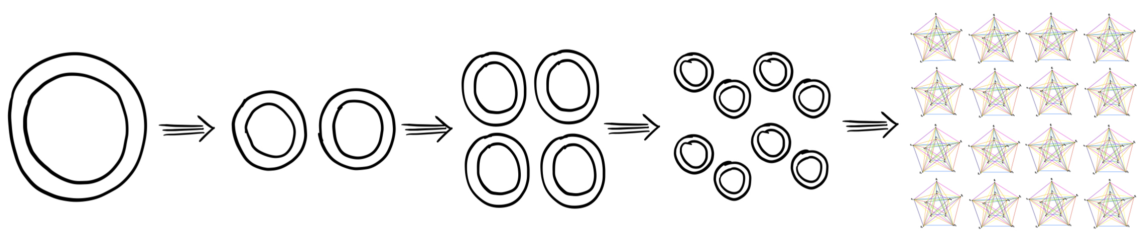
However, this solution has rounds in stage 1 with no special couples dancing, which may be both confusing, and not as pretty for the audience. But, there is a different solution! Notice that after completing the standard waltz mixer, we can divide the dancers into two groups that never have to interact again. Each of the groups has not danced within itself at all. So, if the number of couples is even, we can think of each of these groups as a fresh set of dancers that need to do a complete waltz mixer. In essence, we reduced the problem to the same one, but with half as many couples. We can keep doing this until the number of couples is odd, and then proceed as with Construction 2.

Construction 3: Let there be $2n$ dancers. Let 2^k be the largest power of 2 that divides n, so $n = m2^k$ where m is odd. This mixer will have three stages.

In stage 0, we divide our dancers into 2 groups, perform the standard waltz mixer, and then split off entirely. This is repeated k times until we have 2^k groups of m couples.

Stages 1 and 2 are the same as in Construction 2, performed by each of the 2^k groups. $\triangleleft$

It's hard to find enough waltz dancers and a long enough song for this construction to become practical, but it's still fun to think about 160 dancers splitting into 16 groups of 10 like a fractal, and then completing with 16 separate performances of stages 1 and 2.



This way, also, there is a special couple during each round of stage 2. Again, the order of the round can be completely rearranged, so its possible to instead start with 16 individual groups and then merge into one giant one.

5 Conclusion

In this text we saw how to create a waltz mixer where everyone dances with everyone. Our first solution was clean, and worked for any amount of couples, but it singled out one dancer. Our second solution didn't single out a dancer, but only worked for odd n. Later, we saw a couple options for how to extend this to even n as well.

There are a few other solutions that I did not mention in this text because they are not even close to practical. See if you can think of some yourself. What if it took 3 people to waltz instead of 2 and we want everyone to dance with everyone? What if our dancers want to dance a specific number of rounds with each person that depends on the person? When is that possible? There are many more questions to answer.


How do we do this, guys?