An Intro to Fourier Series

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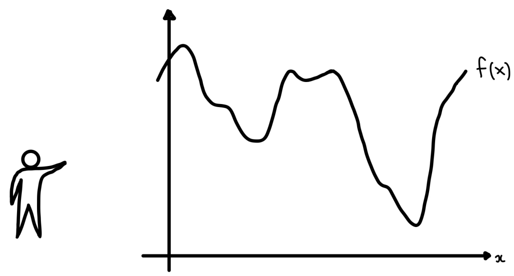
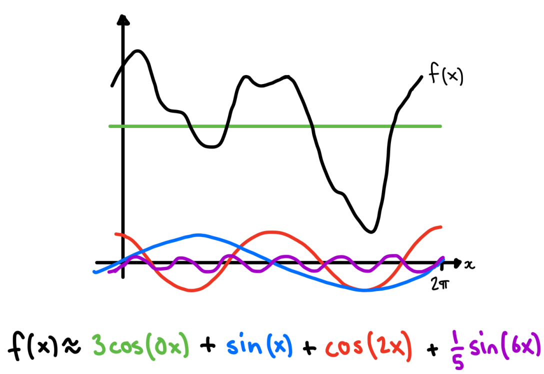
1 Introduction

Imagine we have some function f that we want to work with.



However, we only know how to work with the simplest of trigonometric wave functions—sines and cosines of the form $c\sin(nx)$ or $c\cos(nx)$. What do we do? A natural idea is to try to represent f as a bunch of these sines and cosines. Fortunately, it turns out we can do this for many functions—we can approximate any nice periodic function pretty well as a sum of carefully picked sines and cosines. It has to be periodic because all functions of the form $c\sin(nx)$ or $c\cos(nx)$ are periodic so their sum will also be periodic.



This is the essence of Fourier series. In this paper, we will make this idea rigorous. We'll explore what we mean by "nice" and which wave functions to pick. We will prove that the approximation we have is good in some sense and look at how we can make it even better. In the end, we will see why Fourier series are relevant in the real world.

To get the most out of this paper, an understanding of limits, differentiabilty and continuity of functions, integrals, and basic trigonometry is needed.

2 Definitions and Preliminaries

Let's start with a few definitions and preliminaries that we will use throughout the rest of the paper.

2.1 Trigonometric Formulas

We will use the following formulas without proof. The Wikipedia page on Proofs of Trigonometric Identities has good explanations for why they are true.

(i) Sine of a sum:

$$\sin(a+b) = \sin(a)\cos(b) + \sin(b)\cos(a)$$

(ii) Cosine of a difference:
$$\cos(a-b) = \sin(a)\sin(b) + \cos(a)\cos(b)$$

(iii) Product of sine and cosine:
$$\cos(a)\sin(b) = \frac{\sin(a+b) - \sin(a-b)}{2}$$

(iv) Product of sine and sine:
$$\sin(a)\sin(b) = \frac{\cos(a-b) - \cos(a+b)}{2}$$

(v) Half angle of sine:
$$\sin^2(a/2) = \frac{1-\cos(a)}{2}$$

2.2 Integrals

There will be a few times when we take an integral of the same function over two different intervals. To make the proofs cleaner, we will write
$$\left(\int_a^b + \int_c^d\right) f(x)dx$$
to denote
$$\int_a^b f(x)dx + \int_c^d f(x)dx.$$

2.3 Function Properties

The following proerties will be useful later on.

Definition 1: A function is said to be *uniformly continuous* if for any ε, there exists a δ such that
$$|f(x) - f(y)| < \varepsilon$$
for any x, y where $|x-y| < \delta$. ◁

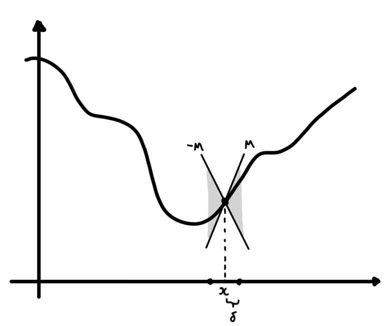
This is essentially the same thing as continuous, but the δ does not depend on x. Any uniformly continuous function is continuous and any continuous function on a closed interval is uniformly continuous. We will use this fact later without proof. A nice proof can be found on Proof Wiki.

Definition 2: A function is said to be *p-periodic* if $f(x+p) = f(x)$ for all x. ◁

Intuitively, this means f repeats itself in intervals of length p. In this paper, we will look specifically 2π-periodic functions.

Definition 3: A function is said to satisfy the *Lipschitz condition* at x if there exist real numbers $M, \delta > 0$ such that
$$|f(x) - f(y)| \le M|x-y| \quad \text{when} \quad |x-y| < \delta.$$
◁

Intuitively, we can think of this as being able to draw two lines of slope M and $-M$ going through x and have f be entirely contained within the left and right sections within the δ-range of x.



3 Fourier Series

Now, let's define the Fourier series of a function. Since we are working with 2π-periodic functions, we only need to examine values of x in any window of size 2π. In this paper, we will work with $x \in [-\pi, \pi]$.

Definition 4: Let f be a 2π-periodic function. The *Fourier series* of f is

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \big(a_n \cos(nx) + b_n \sin(nx)\big),$$

where

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos(nt) dt$$

and

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin(nt) dt.$$

Here, a_n and b_n are called *Fourier coefficients* for f. We can also look at the partial sums instead of the infinite series. We denote them by

$$S_N(x) = \frac{a_0}{2} + \sum_{n=1}^{N} \big(a_n \cos(nx) + b_n \sin(nx)\big).$$

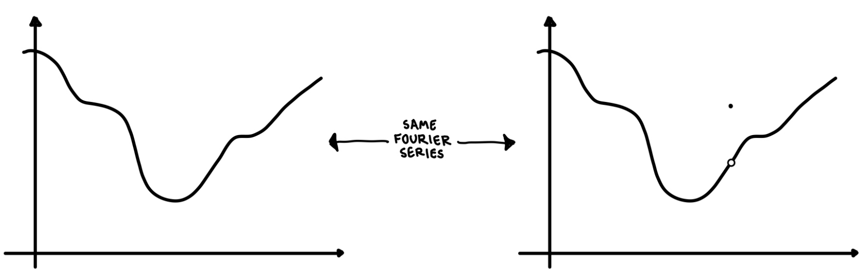
◁

We see that the partial sums S_N are just a sum of many sines and cosines, like we wanted (plus the constant term $a_0/2$). Each trigonometric function is either of the form $a_n \cos(nx)$ or $b_n \sin(nx)$. Our goal is to show that these partial sums approximate f well. This leads us to convergence.

3.1 Convergence

Ultimately, we want to show that S_N converges to f. But what does it mean for a sequence of functions to converge to another function? One answer to this question is *pointwise convergence*. Namely, S_N converges to f pointwise, if for any x, $S_N(x)$ converges to f.

Unfortunately, S_N doesn't converge to f for all functions. Here's one way to see this. Let's take any function f. If we change the value of f at one point, none of the Fourier coefficients change, so S_N also do not change, and the partial sums can't converge to both the new and the old value at that point.



So, we need to restrict ourselves to "nice" functions. We will show that functions satisfying the Lipschitz condition everywhere are nice enough. We will also show that every differentiable function satisfies the Lipschitz condition everywhere, so differentiable functions are also nice.

Pointwise convergence is not the only way we could have answered that question, however. A stronger property is *uniform convergence*. This is when the maximum difference between S_N and f approaches 0. In other words, $\lim_{N\to\infty} \big(\max_x(|S_N(x) - f(x)|)\big) = 0$. In a sense, this means that we can use the same N for all x, so S_N converges to f at about the same rate over all x. After we discuss pointwise convergence, we will see that we can modify our partial sums a little to obtain uniform convergence as well.

3.2 The Dirichlet Kernel and Pointwise Convergence

To help us prove convergence, we will define something called the Dirichlet kernel. It is closely related to the partial sums $S_N(x)$ and is easier to work with.

Definition 5: The *N-th Dirichlet kernel* denoted by $D_N(x)$ is the function

$$D_N(x) = \frac{1}{2} + \sum_{n=1}^{N} \cos(nx).$$

◁

The following lemma connects the Dirichlet kernel to the partial sums and gives us some properties we will use later.

Lemma 1: The N-th Dirichlet kernel D_N has the following properties for $x \in [-\pi, \pi]$.

(i) Let S_N be the Fourier partial sums for some periodic function f. Then, the following holds:

$$S_N(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x-t)D_N(t)dt.$$

(ii) For any N,

$$\frac{1}{\pi}\int_{-\pi}^{\pi} D_N(t)dt = 1.$$

(iii) For $\sin(x/2) \neq 0$,

$$D_N(x) = \frac{\sin\left((N+\frac{1}{2})x\right)}{2\sin\left(\frac{x}{2}\right)}.$$

◁

The first point expresses $S_N(x)$ in terms of $D_N(x)$. The second gives us a nice identity we will use. The third expresses $D_N(x)$ differently, without a sum[1].

Proof of Lemma 1: We will use the trigonometric identities for the proof.

(i) We can write out the Fourier coefficients in the definition to see that

$$\begin{aligned}
S_N(x) &= \frac{a_0}{2} + \sum_{n=1}^{N} \left(a_n \cos(nx) + b_n \sin(nx)\right) \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)dt + \frac{1}{\pi}\sum_{n=1}^{N} \left(\cos(nx)\int_{-\pi}^{\pi} f(t)\cos(nt)dt + \sin(nx)\int_{-\pi}^{\pi} f(t)\sin(nt)dt\right) \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)dt + \frac{1}{\pi}\sum_{n=1}^{N} \int_{-\pi}^{\pi} f(t)\left(\cos(nx)\cos(nt) + \sin(nx)\sin(nt)\right) dt \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)dt + \frac{1}{\pi}\sum_{n=1}^{N} \int_{-\pi}^{\pi} f(t)\cos(nx-nt)dt.
\end{aligned}$$

We used the cosine of a difference trigonometric formula for the last step. Since $f(t)\cos(nx-nt)$ is a 2π-periodic function, the integral is the same over any interval of length 2π. So, we can substitute τ

[1] Note that for $\sin(x/2) = 0$, it's relatively easy to show that $D_N(x) = N + 1/2$ because x is a multiple of 2π.

for $x - t$ without changing the integral. We have that

$$\begin{aligned} S_N(x) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-\tau)d\tau + \frac{1}{\pi}\sum_{n=1}^{N} \int_{-\pi}^{\pi} f(x-\tau)\cos(n\tau)d\tau \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x-\tau)\left(\frac{1}{2} + \sum_{n=1}^{N} \cos(n\tau)\right)d\tau \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x-\tau)D_N(\tau)d\tau, \end{aligned}$$

which is what we wanted to show.

(ii) We can use linearity of integrals and the fact that $\int_{-\pi}^{\pi}\cos(nt)dt = 0$ for all n to see that

$$\begin{aligned} \frac{1}{\pi}\int_{-\pi}^{\pi} D_N(t)dt &= \frac{1}{\pi}\int_{-\pi}^{\pi}\left(\frac{1}{2} + \sum_{n=1}^{N}\cos(nt)\right)dt \\ &= \frac{1}{\pi}\left(\int_{-\pi}^{\pi}\frac{1}{2}dt + \sum_{n=1}^{N}\int_{-\pi}^{\pi}\cos(nt)dt\right) \\ &= \frac{1}{\pi}\left(\frac{1}{2}\cdot 2\pi + \sum_{n=1}^{N} 0\right) \\ &= 1. \end{aligned}$$

(iii) We can consider the equation we want to prove with both sides multiplied by $2\sin(x/2)$. Using the product of sine and cosine trigonometric formula, we end up with a telescoping sum. More precisely,

$$\begin{aligned} 2\sin\left(\frac{x}{2}\right)D_N(x) &= \sin\left(\frac{x}{2}\right) + \sum_{n=1}^{N} 2\cos(nx)\sin\left(\frac{x}{2}\right) \\ &= \sin\left(\frac{x}{2}\right) + \sum_{n=1}^{N}\left[\sin\left(nx + \frac{x}{2}\right) - \sin\left(nx - \frac{x}{2}\right)\right] \\ &= \sin\left(Nx + \frac{x}{2}\right). \end{aligned}$$

So, for $\sin(x/2) \neq 0$, we can divide both sides by $2\sin(x/2)$ to get

$$D_N(x) = \frac{\sin\left((N+\frac{1}{2})x\right)}{2\sin\left(\frac{x}{2}\right)}.$$

With this, we have proven all three points of Lemma 1. □

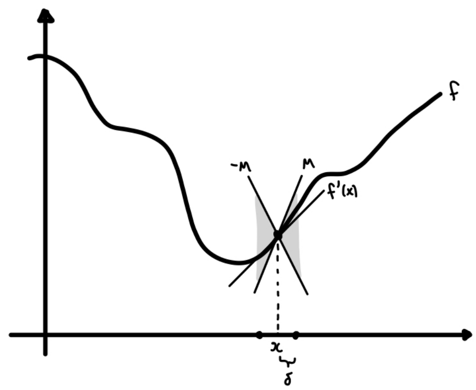
Remember that our first goal with Fourier series is to prove the following.

Theorem 1 (Pointwise Convergence): Let $f(x)$ be a continuously differentiable 2π-periodic function. Let S_N be the Fourier partial sums for f. Then, for any $x \in [-\pi, \pi]$, we have that $S_N(x)$ converges to $f(x)$. ◁

The first step is to show that every continuously differentiable function satisfies a Lipschitz condition everywhere.

Lemma 2: If f is continuous and differentiable at x, then f satisfies a Lipschitz condition at x. ◁

If we recall the intuition behind the definition of the Lipschitz condition, we can imagine selecting M to be more than whatever the derivative is. Then, it makes sense that for some small δ-range, the function would be entirely within the left and right sections.



Let's prove it formally.

Proof of Lemma 2: By the definition of differentiability, we have that

$$\lim_{y \to x} \frac{f(y) - f(x)}{y - x} = f'(x).$$

Let's set $\varepsilon = 1$. By the definition of convergence, we know there exists some $\delta > 0$ such that if $|y - x| < \delta$, then

$$\left| \frac{f(y) - f(x)}{y - x} - f'(x) \right| < \varepsilon = 1.$$

By the triangle inequality,

$$\left| \frac{f(y) - f(x)}{y - x} \right| - |f'(x)| < 1,$$

so

$$|f(y) - f(x)| < (1 + |f'(x)|) \cdot |y - x|.$$

Setting $M = 1 + |f'(x)|$ satisfies the Lipschitz condition. □

Now, instead of proving Theorem 1, we can prove the following lemma.

Lemma 3: Let $f(x)$ be a 2π-periodic function that satisfies a Lipschitz condition everywhere. Let S_n be the Fourier partial sums for f. Then, for any $x \in [-\pi, \pi]$, we have that $S_n(x)$ converges to $f(x)$. ◁

Let's look at how we will approach this lemma. Using Lemma 1(i), we can write out $S_n(x)$ as

$$\frac{1}{\pi} \int_{-\pi}^{\pi} f(x - t) D_n(t) dt.$$

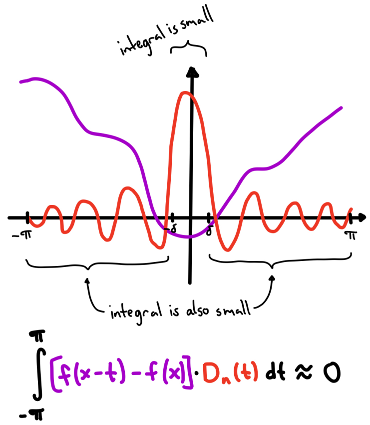
Our goal is to show that this converges to $f(x)$. We can also use lemma 1(ii) to write $f(x)$ as

$$\frac{1}{\pi} \int_{-\pi}^{\pi} f(x) D_n(t) dt.$$

This allows us to look at the difference between S_n and f as one integral. So,

$$S_n(x) - f(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x - t) D_n(t) dt - \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) D_n(t) dt = \frac{1}{\pi} \int_{-\pi}^{\pi} \left[f(x - t) - f(x) \right] D_n(t) dt.$$

We want to show that this converges to 0. To do this, we will split this integral into two parts, the things close to 0 and the things on the edges. More precisely, for any small ε, we will choose some small δ and look at the integral over $[-\delta, \delta]$ separately from the integral over the intervals $[-\pi, -\delta]$ and $[\delta, \pi]$. We will show that no matter what δ we choose, the outsides converge to 0, and then we'll pick a δ small enough for the middle to be very small.



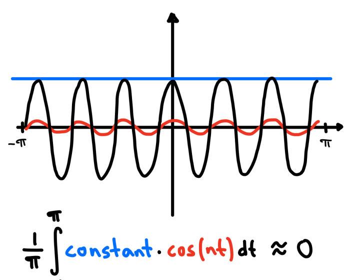
Let's start with showing that the outsides converge to 0 no matter how small δ is. This is called the Riemann Localization Theorem. The "localization" comes from the fact that we are localizing our problem to a small interval around 0 by saying that the outsides are small. We prove it in two steps. First we'll prove the Riemann-Lebesgue Lemma, which we will then use as a tool to prove the Riemann Localization Theorem.

Lemma 4 (Riemann-Lebesgue): Let f be a function that's integrable over the interval $[-\pi, \pi]$. Then,

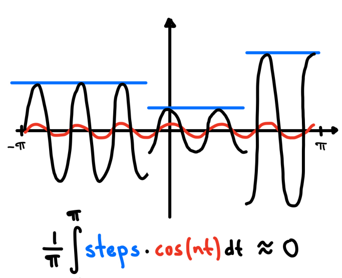
$$\lim_{n\to\infty} \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)dt = 0 = \lim_{n\to\infty} \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)dt.$$

In other words, the Fourier coefficients approach 0 as n increases. ◁

Intuitively, we are multiplying a nice function by a wave-like function and calculating the integral with narrower and narrower waves. If f were constant, it would make sense that the integral approaches 0 because about half of the waves are negative.



A similar situation happens if f is a step function.



We can approximate any integrable function with a step function, so the result should hold for all integrable functions. Let's write this out formally.

Proof of Lemma 4: Let $\varepsilon > 0$. Since f is integrable on $[-\pi, \pi]$, there is a partition $t_0, t_1, \ldots, t_k$ where $t_0 = -\pi$ and $t_k = \pi$ such that

$$0 \leq \int_{-\pi}^{\pi} f(t)dt - \sum_{i=1}^{k} m_i(t_i - t_{i-1}) < \frac{\varepsilon}{2},$$

where m_i is the minimum of f on the interval $[t_{i-1}, t_i]$. Now, define $g(t)$ to be the step function $\sum_{i=1}^{k} m_i \chi[t_{i-1}, t_i]$. Then the above inequality is

$$0 \leq \int_{-\pi}^{\pi} (f(t) - g(t))dt < \frac{\varepsilon}{2}$$

We can use the triangle inequality and the fact that $0 \leq \int_{-\pi}^{\pi} (f(t) - g(t))dt$ to see that

$$\begin{aligned}
\left| \int_{-\pi}^{\pi} f(t)\cos(nt)dt \right| &\leq \left| \int_{-\pi}^{\pi} g(t)\cos(nt)dt \right| + \left| \int_{-\pi}^{\pi} \big(f(t) - g(t)\big)\cos(nt)dt \right| \\
&\leq \left| \int_{-\pi}^{\pi} g(t)\cos(nt)dt \right| + \left| \int_{-\pi}^{\pi} \big(f(t) - g(t)\big)\, dt \right| \\
&\leq \left| \int_{-\pi}^{\pi} g(t)\cos(nt)dt \right| + \frac{\varepsilon}{2} \\
&= \left| \sum_{i=1}^{k} \int_{x_{i-1}}^{x_i} m_i \cos(nt)dt \right| + \frac{\varepsilon}{2} \\
&= \left| \frac{1}{n} \sum_{i=1}^{k} \big(m_i \sin(nx_i) - m_i \sin(nx_{i-1})\big) \right| + \frac{\varepsilon}{2}.
\end{aligned}$$

So, if we select n large enough that

$$\left| \frac{1}{n} \sum_{i=1}^{k} \big(m_i \sin(nx_i) - m_i \sin(nx_{i-1})\big) \right| < \frac{\varepsilon}{2},$$

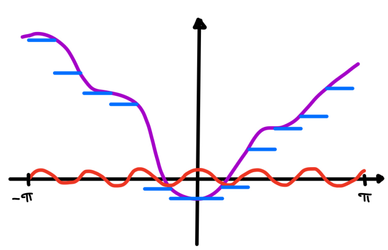
then

$$\left| \int_{-\pi}^{\pi} f(t)\cos(nt)dt \right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

and we are done. The same exact reasoning works to show that

$$\lim_{n\to\infty} \frac{1}{\pi} \int_{-\pi}^{\pi} f(t)\sin(nt)dt = 0.$$

□



$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cdot\cos(nt)\,dt \approx \frac{1}{\pi}\int_{-\pi}^{\pi} \text{steps}\cdot\cos(nt)\,dt \approx 0$$

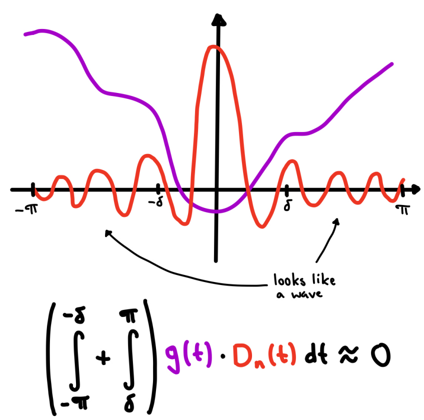
Now let's use Lemma 4 as a tool in the second step.

Lemma 5 (Riemann Localization Theorem): Let g be a function that's integrable over the interval $[-\pi, \pi]$ and pick any δ so that $0 < \delta \leq \pi$. Then,

$$\lim_{n\to\infty}\left[\left(\int_{-\pi}^{-\delta}+\int_{\delta}^{\pi}\right)g(t)D_n(t)dt\right]=0$$

◁

The Dirichlet kernel also somewhat resembles a wave on the outsides of the interval $[-\delta, \delta]$, so the statement of the Riemann Localization Theorem is similar to that of the Riemann-Lebesgue Lemma. We are multiplying an integrable function by something that looks like a wave and expecting the integral to approach 0 as n grows.



We can't apply the Riemann-Lebesgue Lemma directly because the wave-like function there was a sine or a cosine, and here we have something more complicated. However, if we express the Dirichlet kernel as a linear combination of sines and cosines, then we can apply it. So let's do that more formally.

Proof of Lemma 5: We can use Lemma 1(iii) and the sine of a sum trigonometric formula to see that

$$\begin{aligned}\lim_{n\to\infty}\left[\left(\int_{-\pi}^{-\delta}+\int_{\delta}^{\pi}\right)g(t)D_n(t)dt\right] &= \lim_{n\to\infty}\left[\left(\int_{-\pi}^{-\delta}+\int_{\delta}^{\pi}\right)g(t)\frac{\sin\left((n+\frac{1}{2})t\right)}{2\sin\left(t/2\right)}dt\right]\\ &= \lim_{n\to\infty}\left[\left(\int_{-\pi}^{-\delta}+\int_{\delta}^{\pi}\right)g(t)\frac{\sin(nt)\cos\left(t/2\right)+\sin\left(t/2\right)\cos(nt)}{2\sin\left(t/2\right)}dt\right]\\ &= \lim_{n\to\infty}\left[\frac{1}{2}\left(\int_{-\pi}^{-\delta}+\int_{\delta}^{\pi}\right)g(t)\Big(\sin(nt)\cot\left(t/2\right)+\cos(nt)\Big)\,dt\right]\end{aligned}$$

Now, if we let

$$g_1(t)=\begin{cases}0 & \text{if } -\delta<t<\delta\\ g(t) & \text{if } \delta\leq|t|\leq\pi\end{cases}$$

and

$$g_2(t)=\begin{cases}0 & \text{if } -\delta<t<\delta\\ g(t)\cot(t/2) & \text{if } \delta\leq|t|\leq\pi\end{cases},$$

then we can plug in g_1 and g_2 into our last limit above. Since both g_1 and g_2 are 0 over the interval $[-\delta, \delta]$, we can then extend the integral to the entire interval $[-\pi, \pi]$. So, we have

$$\begin{aligned}
\lim_{n\to\infty}\left[\left(\int_{-\pi}^{-\delta}+\int_{\delta}^{\pi}\right)g(t)D_n(t)dt\right] &= \lim_{n\to\infty}\left[\frac{1}{2}\left(\int_{-\pi}^{-\delta}+\int_{\delta}^{\pi}\right)\big(g_2(t)\sin(nt)+g_1(t)\cos(nt)\big)\,dt\right] \\
&= \lim_{n\to\infty}\left[\frac{1}{2}\int_{-\pi}^{\pi}\big(g_2(t)\sin(nt)+g_1(t)\cos(nt)\big)\,dt\right] \\
&= \lim_{n\to\infty}\left[\frac{1}{2}\int_{-\pi}^{\pi}g_2(t)\sin(nt)dt\right]+\lim_{n\to\infty}\left[\frac{1}{2}\int_{-\pi}^{\pi}g_1(t)\cos(nt)dt\right] \\
&= 0+0=0
\end{aligned}$$

by the Riemann-Lebesgue Lemma (Lemma 4). Note that it's crucial that both g_1 and g_2 are integrable (since they are made from pieces of integrable functions). □

Remember that our goal is to show that

$$\frac{1}{\pi}\int_{-\pi}^{\pi}\big[f(x-t)-f(x)\big]\,D_N(t)dt$$

converges to 0 as N grows. The Riemann Localization Theorem essentially lets us throw away the sides of the interval and only consider

$$\frac{1}{\pi}\int_{-\delta}^{\delta}\big[f(x-t)-f(x)\big]\,D_N(t)dt$$

for our choice of delta. The key here is that we can choose δ to be so small, that the integral over $[-\delta, \delta]$ is also small. Let's formalize all of our discussion to complete the proof of pointwise convergence.

Proof of Lemma 3: Let $f(x)$ be a 2π-periodic function that satisfies a Lipschitz condition everywhere. Let $S_n(x)$ the the Fourier partial sums for f. Let x be some point on the interval $[-\pi, \pi]$. As in our discussion after the statement of Lemma 3,

$$S_n(x)-f(x)=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x-t)D_n(t)dt-\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)D_n(t)dt=\frac{1}{\pi}\int_{-\pi}^{\pi}\big[f(x-t)-f(x)\big]\,D_n(t)dt.$$

Let $\varepsilon > 0$. We will find an N such that for any $n > N$, we have $|S_n(x) - f(x)| < \varepsilon$. Let us bound the middle section of the interval. First, notice that

$$\lim_{t\to 0}\frac{t}{2\sin(t/2)}=1$$

by L'Hôpital's rule. So, there exists some bound k and some positive $\delta_1 < \pi$ such that

$$\left|\frac{t}{2\sin(t/2)}\right|<k$$

when $0 < |t| < \delta_1$. Since $\sin(x) \le 1$, this means

$$|tD_n(t)|=\left|\frac{t}{2\sin(t/2)}\cdot\sin\big((n+1/2)t\big)\right|<k$$

when $0 < |t| < \delta_1$. When $t = 0$, then $|tD_n(t)| = 0 < k$ as well, so

$$|tD_n(t)|<k$$

on the entire interval $[-\delta_1, \delta_1]$. Since f satisfies a Lipschitz condition at x, there exists some $\delta_2 > 0$ and bound M such that

$$|f(x)-f(t)|\le M|x-t|$$

when $|x-t| < \delta_2$.

Now, let's pick $\delta = \min(\delta_1, \delta_2, \varepsilon/(4Mk)$. Then, for any n, by the triangle inequality,

$$\begin{aligned}\left|\frac{1}{\pi}\int_{-\delta}^{\delta}\left[f(x-t)-f(x)\right]D_n(t)dt\right| &\leq \int_{-\delta}^{\delta}|f(x-t)-f(x)||D_n(t)|dt \\ &\leq \int_{-\delta}^{\delta} M|t||D_n(t)|dt \\ &\leq \int_{-\delta}^{\delta} Mkdt \\ &= 2Mk\delta \leq \frac{\varepsilon}{2}.\end{aligned}$$

Note that we used that $\delta \leq \delta_1$, that $\delta \leq \delta_2$, and that $\delta \leq \varepsilon/(4Mk)$ in the above inequalities. This shows that the middle part of the integral can be made small. By the Riemann Localization Theorem (Lemma 5), there exists some N such that for any $n > N$,

$$\left|\left(\int_{-\pi}^{-\delta} + \int_{\delta}^{\pi}\right)\left[f(x-t)-f(x)\right]D_n(t)dt\right| < \frac{\varepsilon}{2}.$$

That shows that the outsides of the integral are small. Combining these, we have that

$$\begin{aligned}|S_n(x)-f(x)| &= \left|\frac{1}{\pi}\int_{-\pi}^{\pi}\left[f(x-t)-f(x)\right]D_n(t)dt\right| \\ &\leq \left|\frac{1}{\pi}\int_{-\delta}^{\delta}\left[f(x-t)-f(x)\right]D_n(t)dt\right| + \left|\left(\int_{-\pi}^{-\delta} + \int_{\delta}^{\pi}\right)\left[f(x-t)-f(x)\right]D_n(t)dt\right| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

So, $S_n(x)$ converges to $f(x)$ and we are done. □

We can finally conclude with our result about pointwise convergence.

Proof of Theorem 1: Any continuously differentiable function satisfies a Lipschitz condition everywhere by Lemma 2. The Fourier partial sums of any function satisfying a Lipschitz condition everywhere converge to that function by Lemma 3. □

3.3 The Fejér Kernel and Uniform Convergence

We saw in the previous section that Fourier sums converge pointwise for nice functions. Unfortunately, the same doesn't hold if we want uniform convergence. However, if we consider the Cesáro sums of Fourier partial sums instead, then we actually can achieve uniform convergence to any continuous function.

Definition 6: Let $\{a_0, a_1, a_2, \dots\}$ be a sequence. The N*-th Cesáro sum* of the sequence is

$$\frac{1}{N+1}\sum_{n=0}^{N} a_n.$$

◁

Note: We are starting our sequence with index 0 in this definition only because it will be convenient for us later.

Intuitively, Cesáro sums are simply the average of the first few terms.

Definition 7: We denote the N-th cesáro sums of the Fourier partial sums by

$$\sigma_N(x) = \frac{1}{N+1}\sum_{n=0}^{N} S_n(x)$$

◁

We created the Dirichlet kernel to help us work with S_N in the previous section. To work with σ_N, we need to create an analogous tool. This motivates the following definition.

Definition 8: The *N-th Fejér kernel* denoted by $F_N(x)$ is the N-th Cesáro sum of the Dirichlet kernels:

$$F_N(x) = \frac{1}{N+1} \sum_{n=0}^{N} D_N(x).$$

◁

Just like with the Dirichlet kernel, there are a few very similar useful properties we can prove.

Lemma 6: The N-th Fejér kernel F_N has the following properties for $x \in [-\pi, \pi]$.

(i) Let σ_N be the Cesáro sum of the Fourier partial sums for some periodic function f. Then, the following holds:

$$\sigma_N(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x-t) F_N(t) dt.$$

(ii) For any N,

$$\frac{1}{\pi} \int_{-\pi}^{\pi} F_N(t) dt = 1.$$

(iii) For $\sin(x/2) \neq 0$,

$$F_N(x) = \frac{1}{2(N+1)} \left(\frac{\sin\left(\frac{(n+1)x}{2}\right)}{\sin\left(\frac{x}{2}\right)} \right)^2.$$

Note that this means that the Fejér kernel is always nonnegative. ◁

Proof of Lemma 6:

(i) By Lemma 1(i),

$$\begin{aligned}
\sigma_N(x) &= \frac{1}{N+1} \sum_{n=0}^{N} S_n(x) \\
&= \frac{1}{N+1} \sum_{n=0}^{N} \frac{1}{\pi} \int_{-\pi}^{\pi} f(x-t) D_N(t) dt \\
&= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x-t) \left(\frac{1}{N+1} \sum_{n=0}^{N} D_N(t) \right) dt \\
&= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x-t) F_N(t) dt.
\end{aligned}$$

(ii) By Lemma 1(ii),

$$\frac{1}{\pi} \int_{-\pi}^{\pi} F_N(t) dt = \frac{1}{N+1} \sum_{n=0}^{N} \frac{1}{\pi} \int_{-\pi}^{\pi} D_N(t) dt = \frac{1}{N+1} \cdot (N+1) = 1.$$

(iii) We can use the product of sine and sine trigonometric formula and get a telescoping sum. Then, we can use the half angle formula to bring the expression to the right form. By Lemma 1(iii),

$$\begin{aligned}
F_N(x) &= \frac{1}{N+1}\sum_{n=0}^{N} D_N(x) \\
&= \frac{1}{N+1}\sum_{n=0}^{N} \frac{\sin\left(Nx+\frac{x}{2}\right)}{2\sin\left(\frac{x}{2}\right)} \\
&= \frac{1}{2(N+1)\left(\sin(x/2)\right)^2}\cdot\sum_{n=0}^{N}\sin\left(Nx+\frac{x}{2}\right)\sin\left(\frac{x}{2}\right) \\
&= \frac{1}{2(N+1)\left(\sin(x/2)\right)^2}\cdot\sum_{n=0}^{N}\frac{\cos(nx)-\cos\left((n+1)x\right)}{2} \\
&= \frac{1}{2(N+1)\left(\sin(x/2)\right)^2}\cdot\frac{1-\cos\left((N+1)x\right)}{2} \\
&= \frac{1}{2(N+1)\left(\sin(x/2)\right)^2}\cdot\sin^2\left((N+1)\frac{x}{2}\right) \\
&= \frac{1}{2(N+1)}\left(\frac{\sin\left(\frac{(n+1)x}{2}\right)}{\sin\left(\frac{x}{2}\right)}\right)^2.
\end{aligned}$$

□

Now, with these tools, we can actually approach uniform convergence.

Theorem 2 (Uniform Convergence): Let $f(x)$ be a continuous and 2π-periodic function. Let σ_n be the Cesáro sums of the Fourier partial sums for f. Then, $\sigma_n(x)$ converges to $f(x)$ uniformly as n grows. ◁

Surprisingly, the proof of this theorem is a lot more straightforward than that of pointwise convergence. Since the Fejér kernel is nonnegative, we can expand the interval over which we are integrating and the integral will only grow. This helps us avoid the suffering with the convergence of the outsides.

Proof of Theorem 2: Let $\varepsilon > 0$. We will show that there exists an N such that for any $n > N$, we have $|\sigma_n(x) - f(x)| < \varepsilon$ for all x (where N doesn't depend on x).

Since f is continuous on a closed interval, it must be bounded and uniformly continuous. So there exists an M such that $|f(x)| < M$ for all x and there exists a δ such that $0 < \delta \leq \pi$ and

$$|f(x) - f(t)| < \frac{\varepsilon}{2}$$

when $|x - y| < \delta$.

Now let's bound $F_n(t)$. Note that for $\delta \leq |t| \leq \pi$, we have $\sin^2(\delta/2) \leq \sin^2(t/2)$. So, by Lemma 6(iii),

$$F_n(t) = \frac{1}{2(n+1)}\left(\frac{\sin\left(\frac{(n+1)t}{2}\right)}{\sin\left(\frac{t}{2}\right)}\right)^2 \leq \frac{1}{2(n+1)}\left(\frac{1}{\sin\left(\frac{t}{2}\right)}\right)^2 \leq \frac{1}{2(n+1)}\left(\frac{1}{\sin\left(\frac{\delta}{2}\right)}\right)^2.$$

This means that if $\delta \leq |t| \leq \pi$, we can choose N large enough that for any $n > N$,

$$F_n(t) < \frac{\varepsilon}{8M}.$$

Let's choose that N. Similar to our proof of pointwise convergence, we can use Lemma 6(i) and 6(ii) and the triangle inequality to see that

$$\begin{aligned}|\sigma_n(x) - f(x)| &= \left|\frac{1}{\pi}\int_{-\pi}^{\pi} f(x-t)F_n(t)dt - \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)F_n(t)dt\right| \\ &= \left|\frac{1}{\pi}\int_{-\pi}^{\pi} \big[f(x-t) - f(x)\big]\, F_n(t)dt\right| \\ &\leq \frac{1}{\pi}\int_{-\pi}^{\pi} |f(x-t) - f(x)|F_n(t)dt.\end{aligned}$$

Note that we do not need to put an absolute value around $F_n(t)$ because $F_n \geq 0$. We will split this into an integral over $[-\delta, \delta]$ and an integral over $[-\pi, -\delta]$ and $[\delta, \pi]$ like we did in proving pointwise convergence, but this time the bounding will be easier.

If $|t| \leq \delta$, then $|(x-t) - x| \leq \delta$, so uniform continuity gives us that $|f(x-t) - f(x)| < \frac{\varepsilon}{2}$, so

$$\begin{aligned}\frac{1}{\pi}\int_{-\delta}^{\delta} |f(x-t) - f(x)|F_n(t)dt &\leq \frac{\varepsilon}{2}\cdot\frac{1}{\pi}\int_{-\delta}^{\delta} F_n(t)dt \\ &\leq \frac{\varepsilon}{2}\cdot\frac{1}{\pi}\int_{-\pi}^{\pi} F_n(t)dt \\ &= \frac{\varepsilon}{2}.\end{aligned}$$

We were able to expand the interval of integration because $F_n(t) \geq 0$, and the last step comes from Lemma 6(ii).

If $\delta \leq |t| \leq \pi$, then by our choice of N, for any $n > N$, we have $F_n(t) < \frac{\varepsilon}{8M}$. Also by our bound, $|f(x-t) - f(x)| < 2M$ so

$$\begin{aligned}\frac{1}{\pi}\left(\int_{-\pi}^{-\delta} + \int_{\delta}^{\pi}\right)|f(x-t) - f(x)|F_n(t)dt &< 2M\cdot\frac{1}{\pi}\left(\int_{-\pi}^{-\delta} + \int_{\delta}^{\pi}\right)F_n(t)dt \\ &< 2M\cdot\frac{1}{\pi}\left(\int_{-\pi}^{-\delta} + \int_{\delta}^{\pi}\right)\frac{\varepsilon}{8M}dt \\ &< 2M\cdot\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{\varepsilon}{8M}dt \\ &= 2M\cdot\frac{1}{\pi}\cdot 2\pi\cdot\frac{\varepsilon}{8M} = \frac{\varepsilon}{2}.\end{aligned}$$

Combining the two cases, we see that

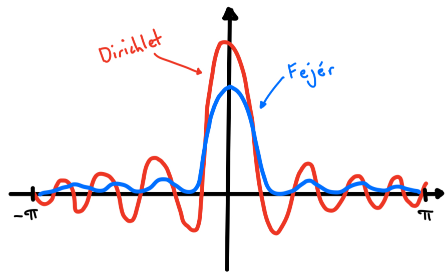
$$|\sigma_n(x) - f(x)| \leq \frac{1}{\pi}\int_{-\pi}^{\pi}|f(x-t) - f(x)|F_n(t)dt < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

So, σ_n converges to f uniformly. □

3.4 Conclusion

So, what did we just do? We proved two main statements. First, the Fourier partial sums of a differentiable function converge to the function pointwise. Second, the Cesáro sums of the Fourier partial sums of a continuous function converge to the function uniformly. This is a little strange, however. The second statement seems significantly stronger than the first. After all, everything that converges uniformly also converges pointwise, and every differentiable function is also continuous. But the proof of the second statement seemed simpler—we didn't have to bother with any localization lemmas. Couldn't we just use the same proof to show that Fouriers partial sums using the Dirichlet kernel?

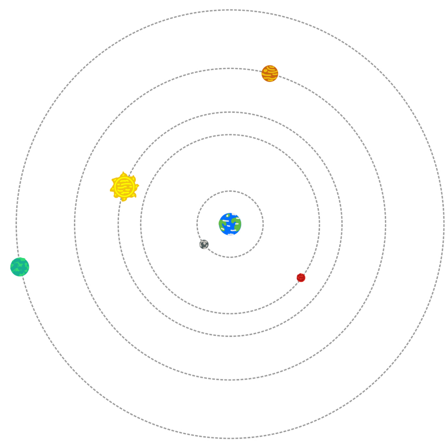
Not exactly. The key difference is that the Fejér kernel is nonnegative. This allows us to forget the absolute value signs around the Fejeŕ kernel. It also lets us bound the integral over the small interval in the center by the integral over the entire interval $[-\pi, \pi]$. The same wouldn't work with the Dirichlet kernel.



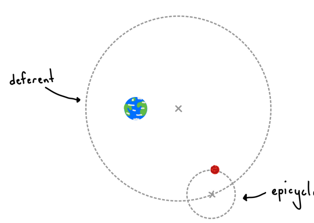
4 Applications Of Fourier Series

In the introduction, we talked about the motivation for studying Fourier series, so let's see how we can use them in real life.

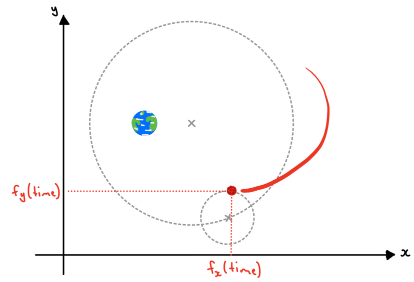
One straightforward example is the Greeks' idea of orbits. As some of us may know, the idea at the time was that everything in the solar system revolves around the earth.



However, astronomers saw that the lengths of seasons did not match up with perfectly circular orbits, and planets weren't rotating around the earth at a constant speed like the stars were. So, to make up for that, they created the concept of a deferent and an epicycle. The deferent is an off-centered circular orbit around the earth and the epicycle is a circular orbit whose center moves along the deferent. The plantets then move along their epicycle.

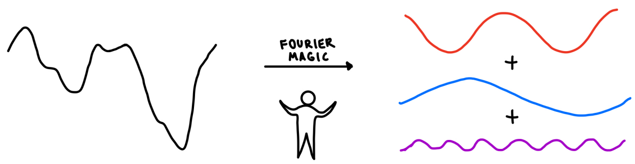


The x and y coordinates of a circular orbit follow a sine and cosine wave as a function of time. Placing the center of one orbit along another orbit is equivalent to adding the x and y coordinates. So, in a sense, the Greek astronomers were trying to represent a function (the planet's true relative path) as a sum of sines and cosines.



Unfortunately, however, the elliptical and heliocentric model of the solar system was developed before the study of Fourier series could have helped the astronomers out.

Outside of planetary orbits, the idea of trying to approximate a function with sines and cosines usually takes the following form. We are given a sum of sines and cosines and we want to figure out what the sines and cosines are.

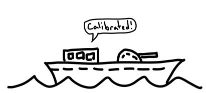


We can see how Fourier series can help us out with this. In reality, the process of converting from the sum of sines and cosines to a list of what the coefficients of the sines and cosines actually are uses something called the Fourier transform, which we won't discuss in this paper. Given that ability, though, we can examine individual waves instead of the mess that a sum of waves can be.

Consider an audio track with a pesky humming noise in the background. If we can split it up into a sum of individual frequencies, then we can simply take out those corresponding to the humming noise and get a cleaner sound. This is how noise supression works.

Seismometers measure the movement of the ground, but the output looks like random vibrations. If we convert it to a sum of individual waves, then we can extract valuable information about the origins of earthquakes.

Parsing the shaking effect of ocean waves on battleships into individual periodic movements can help calibrate the guns so that the angles are accurate.



We can see that Fourier series are useful. This paper cannot cover all aspects of Fourier analysis, but hopefully it gives general insight about their importance and the mathematics behind them.

Sources:

I owe my ability to write this paper to the following sources:

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